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Maximal convergence theorems for functions of squared modulus holomorphic type in \mathbb{R}^2 and some applications

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Abstract

In this paper we extend the theory of maximal convergence introduced by Walsh to functions of squared modulus holomorphic type, i.e.

$$F(x, y) = |g(x + iy)|^2, \quad (x, y) \in L, \quad L \subset \mathbb{R}^2 \text{ compact,}$$

where g is holomorphic in an open connected neighborhood of $\{x + iy \in \mathbb{C} : (x, y) \in L\}$. We introduce in accordance to the well-known complex maximal convergence number for holomorphic functions a real maximal convergence number for functions of squared modulus holomorphic type and prove several maximal convergence theorems. We achieve that the real maximal convergence number for F is always greater or equal than the complex maximal convergence number for g and equality occurs if L is a closed disk in \mathbb{R}^2 . Among other various applications of the resulting approximation estimates we show that for functions F of squared modulus holomorphic type which have no zeros in $\overline{B}_{2,r} := \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} \leq r\}$ the relation

$$\lim_{n \rightarrow \infty} \sup \sqrt[n]{E_n(\overline{B}_{2,r}, F)} = \lim_{n \rightarrow \infty} \sup \sqrt[n]{E_n(\partial B_{2,r}, F)}$$

is valid, where $E_n(\overline{B}_{2,r}, F) := \inf \{\max_{(x,y) \in \overline{B}_{2,r}} |F(x, y) - P_n(x, y)| : P_n : \mathbb{R}^2 \rightarrow \mathbb{R} \text{ a polynomial of degree } \leq n\}$.

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1. Introduction and main results

A central theme in constructive approximation theory is the relation between the smoothness of a function and the speed at which it can be approximated by polynomials. Classical one-dimensional results are for instance theorems of Jackson type [10] and the maximal convergence theorems of Bernstein and Walsh [6,26,25]. Both kind of theorems have attracted much attention and recently some endeavor has been put in to extend them to higher dimensions, e.g. Bernstein–Walsh type theorems for holomorphic functions in \mathbb{C}^N [18,27,19,7], harmonic functions in \mathbb{R}^N [1,2,23], pluriharmonic functions in \mathbb{C}^N [21] and solutions of elliptic equations in \mathbb{R}^N [3,4].

The aim of this work is to extend the theory of maximal convergence, which was introduced by Walsh [25], to functions of squared modulus holomorphic type in \mathbb{R}^2 . We prove several maximal convergence theorems and present various applications of them.

To state maximal convergence results we first have to introduce an approximation measure. Conventionally we choose the n -th polynomial approximation error as follows:

- (i) Let $K \subset \mathbb{C}$ be a compact set and $f : K \rightarrow \mathbb{C}$ be a continuous function. Then the n -th (complex) approximation error is denoted by

$$e_n(K, f) = \inf\{\|f - p_n\|_K, p_n : \mathbb{C} \rightarrow \mathbb{C}, p_n \text{ a polynomial of degree } \leq n\},$$

where $n \in \mathbb{N}$ and $\|\cdot\|_K$ stands for the supremum norm on K .

- (ii) Let $K \subset \mathbb{R}^N$, $N \in \{1, 2\}$, be a compact set and $F : K \rightarrow \mathbb{R}$ be a continuous function. Then the n -th (real) approximation error is defined analogously by

$$E_n(K, F) = \inf\{\|F - P_n\|_K, P_n : \mathbb{R}^N \rightarrow \mathbb{R}, P_n \text{ a polynomial of degree } \leq n\},$$

where $n \in \mathbb{N}$ and $\|\cdot\|_K$ denotes the supremum norm on K .

Now let $f : K \rightarrow \mathbb{C}$ be a continuous function on a compact set $K \subset \mathbb{C}$ such that $\limsup_{n \rightarrow \infty} \sqrt[n]{e_n(K, f)} < 1$ then

$$\rho := \frac{1}{\limsup_{n \rightarrow \infty} \sqrt[n]{e_n(K, f)}} \quad (1.1)$$

is called the (complex) maximal convergence number.

In addition, a sequence $\{p_n\}_{n \in \mathbb{N}}$ of polynomials p_n of degree $\leq n$ is said to converge maximally to f , if for every $R \in (1, \rho)$ the estimate

$$\|f - p_n\|_K \leq \frac{M}{R^n}, \quad n \in \mathbb{N},$$

holds, where $M > 0$ is a constant independent of n .

This terminology will be used analogously for real-valued functions F defined on compact sets in \mathbb{R}^N , $N \in \{1, 2\}$.

Classical maximal convergence results are the so-called Bernstein–Walsh theorems. Bernstein's theorem handles the real case in one dimension.

Theorem 1.1 (Bernstein [6], 1912). *Let $F : [-1, 1] \rightarrow \mathbb{R}$ be continuous and $\rho > 1$. Then*

$$\limsup_{n \rightarrow \infty} \sqrt[n]{E_n([-1, 1], F)} \leq \frac{1}{\rho}$$

if and only if F has a holomorphic extension to the set

$$\{z \in \mathbb{C} : |h(z)| < \rho\},$$

where $h : \mathbb{C} \rightarrow \mathbb{C} \setminus \{z \in \mathbb{C} : |z| < 1\}$ is defined by $h(z) = z + \sqrt{z^2 - 1}$.¹

Walsh and Russell² gave an outstanding generalization of Theorem 1.1 for the complex plane. They showed that the interval $[-1, 1]$ in Theorem 1.1 can be replaced by compact sets $K \subset \mathbb{C}$ whose complement is connected and regular in the sense that for $\hat{\mathbb{C}} \setminus K$, $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, Green's function g_K with pole at infinity exists. Here, Green's function is the uniquely determined function which has a logarithmic singularity at infinity, is continuous in \mathbb{C} , harmonic in $\mathbb{C} \setminus K$ and identically zero on K .

Theorem 1.2 (Walsh [25], 1934). *Let K be a compact subset of \mathbb{C} such that $\hat{\mathbb{C}} \setminus K$ is connected and regular. Furthermore, let $f : K \rightarrow \mathbb{C}$ be continuous and $\rho > 1$. Then*

$$\limsup_{n \rightarrow \infty} \sqrt[n]{e_n(K, f)} \leq \frac{1}{\rho}$$

if and only if $f \equiv \tilde{f}|_K$, where \tilde{f} is a holomorphic function in

$$L_\rho = \{z \in \mathbb{C} : e^{g_K(z)} < \rho\}.$$

An approximation problem of maximal convergence structure for certain *real-analytic functions* in \mathbb{R}^2 was raised in [9]. There it was conjectured that for functions $F : \overline{B}_2 \rightarrow \mathbb{R}$, $\overline{B}_2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$, defined by

$$F(x, y) = \frac{1}{((x - x_0)^2 + (y - y_0)^2)^s}, \quad (1.2)$$

where $s \in (0, \infty)$ and $(x_0, y_0) \in \mathbb{R}^2$ such that $\rho_0 := \sqrt{x_0^2 + y_0^2} > 1$, the relation

$$\limsup_{n \rightarrow \infty} \sqrt[n]{E_n(\overline{B}_2, F)} = \frac{1}{\rho_0} \quad (1.3)$$

holds. In [14] we verified (1.3) by means of Theorem 1.1 and the convexity of best approximants.

The function F in (1.2) can be expressed as the squared modulus of a holomorphic function in some neighborhood of the closed unit disk $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$, i.e.

$$F(x, y) = \frac{1}{((x - x_0)^2 + (y - y_0)^2)^s} = g(z)\overline{g(z)}, \quad z = x + iy,$$

where $g(z) := 1/(z - z_0)^s$, $z_0 = x_0 + iy_0$, is holomorphic in $\mathbb{D}_{\rho_0} := \{z \in \mathbb{C} : |z| < \rho_0\}$. Since g has no holomorphic extension to any neighborhood containing $\overline{\mathbb{D}}_{\rho_0}$, the complex maximal convergence number for g is also ρ_0 by Theorem 1.2. Thus for these particular functions the real maximal convergence number coincides with the complex maximal convergence number.

¹ The branch of the square root is chosen such that $h(x) > 1$ for $x > 1$.

² The generalization of Theorem 1.1 is due to Walsh [24] in the case that $\hat{\mathbb{C}} \setminus K$ is simply connected in $\hat{\mathbb{C}}$ and due to Walsh and Russell [26] if $\hat{\mathbb{C}} \setminus K$ is connected and regular. However, in the literature Theorems 1.1 and 1.2 are just called the Bernstein–Walsh theorems.

In this study we investigate the connection between these two maximal convergence numbers for *arbitrary* functions of squared modulus holomorphic type. More precisely, we will show that for the class of squared modulus holomorphic functions, i.e.

$$F(x, y) = |g(x + iy)|^2, \quad (x, y) \in L, \quad L \subset \mathbb{R}^2 \text{ compact},$$

where g is holomorphic in an open connected neighborhood of $\{x + iy \in \mathbb{C} : (x, y) \in L\}$, the real maximal convergence number for F is always greater or equal than the complex maximal convergence number for g and equality occurs if L is a closed disk in \mathbb{R}^2 .

However, before stating the main results let us bring up some notations.

We abbreviate the disk of radius r and center 0 in \mathbb{R}^2 by

$$B_{2,r} := \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} < r\} \quad \text{in particular } B_2 := B_{2,1},$$

and denote the open disk of radius $r > 0$ about 0 in \mathbb{C} by

$$\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\} \quad \text{especially } \mathbb{D} := \mathbb{D}_1.$$

The annulus A_{r_1, r_2} with center 0 and radii r_1, r_2 is the set

$$A_{r_1, r_2} = \{z \in \mathbb{C} : r_1 < |z| < r_2\}.$$

$\mathcal{H}(G)$ stands for the set of holomorphic functions defined on a domain $G \subset \mathbb{C}$ and $\mathcal{H}(\overline{G})$ for the set of functions holomorphic in some neighborhood of \overline{G} , where \overline{G} is the closure of G and ∂G the boundary of G .

Since we consider functions f defined on sets in \mathbb{C} and functions F defined on sets in \mathbb{R}^2 simultaneously, we will distinguish them for more clarity by small and capital letters.

Theorem 1.3. *Let $g \in \mathcal{H}(\overline{\mathbb{D}}_r)$ and $F : \overline{B}_{2,r} \rightarrow \mathbb{R}$ be given by*

$$F(x, y) = |g(x + iy)|^2.$$

If $\rho \in (1, \infty)$ then the following conditions are equivalent:

- (i) $\limsup_{n \rightarrow \infty} \sqrt[n]{E_n(\overline{B}_{2,r}, F)} = \frac{1}{\rho}.$
- (ii) $g \in \mathcal{H}(\mathbb{D}_{r\rho}) \setminus \mathcal{H}(\overline{\mathbb{D}}_{r\rho}).$

Furthermore,

$$\limsup_{n \rightarrow \infty} \sqrt[n]{E_n(\overline{B}_{2,r}, F)} = 0$$

if and only if g has a holomorphic extension to \mathbb{C} .

We see, in light of that theorem Braess' approximation problem, the establishment of relation (1.3), is just an application to the special function $F(x, y) = ((x - x_0)^2 + (y - y_0)^2)^{-s}$.

The crucial role for Theorem 1.3 plays

Lemma 1.4. *Let $F : \partial B_{2,r} \rightarrow \mathbb{R}$ be a continuous function, $r \in (0, \infty)$ and $\rho > 1$. If*

$$\limsup_{n \rightarrow \infty} \sqrt[n]{E_n(\partial B_{2,r}, F)} \leq \frac{1}{\rho},$$

then the function

$$h_r(z) = F\left(\frac{r}{2}\left(z + \frac{1}{z}\right), \frac{r}{2i}\left(z - \frac{1}{z}\right)\right), \quad z \in \partial\mathbb{D},$$

has a holomorphic extension to the annulus $A_{1/\rho, \rho}$.

The maximal convergence number ρ for F in Theorem 1.3 is obtained by determining the largest disk in \mathbb{C} to which g has an analytic continuation. A different method to characterize the maximal convergence number is described in the next theorem.

Theorem 1.5. Let $F : \overline{B}_{2,r} \rightarrow \mathbb{R}$ be a function with the representation

$$F(x, y) = |g(x + iy)|^2 \quad \text{where } g \in \mathcal{H}(\overline{\mathbb{D}}_r).$$

Then

$$\limsup_{n \rightarrow \infty} \sqrt[n]{E_n(\overline{B}_{2,r}, F)} = \frac{1}{\rho}$$

if and only if for every $s \in (0, 1]$ the function $h_s : \partial\mathbb{D} \rightarrow \mathbb{C}$,

$$h_s(z) := F\left(s\frac{r}{2}\left(z + \frac{1}{z}\right), s\frac{r}{2i}\left(z - \frac{1}{z}\right)\right),$$

has a holomorphic extension to $A_{s/\rho, \rho/s}$, and at least one of these extensions is not holomorphic continuable to any neighborhood of $A_{s/\rho, \rho/s}$.

In particular, to each $\varepsilon > 0$ there exists a number $\hat{s} \in (1 - \varepsilon, 1]$ such that $h_{\hat{s}}$ has no holomorphic extension to any neighborhood of $\overline{A}_{\hat{s}/\rho, \rho/\hat{s}}$.

In Section 3 we discuss several applications and consequences of Theorem 1.3 and the results we have developed in order to prove this theorem. To mention only one example we show that for functions F of squared modulus holomorphic type the relation

$$\limsup_{n \rightarrow \infty} \sqrt[n]{E_n(\overline{B}_{2,r}, F)} = \limsup_{n \rightarrow \infty} \sqrt[n]{E_n(\partial B_{2,r}, F)}$$

is valid if F has no zeros in $B_{2,r}$, whereas the statement fails if F has zeros in $B_{2,r}$.

Maximal convergence results for squared modulus holomorphic functions defined on arbitrary compact sets $K \subset \mathbb{R}^2$ will be treated in Section 4. We will show that the real maximal convergence number is always greater or equal than the corresponding complex maximal convergence number and demonstrate that equality does not hold in general.

2. Proofs of Theorem 1.3, Lemma 1.4 and Theorem 1.5

Theorem 1.3 will be proved in several steps. First of all we establish an upper bound for the approximation error E_n .

Lemma 2.1. Let $F : \overline{B}_{2,r} \rightarrow \mathbb{R}$ be given by

$$F(x, y) = |g(x + iy)|^2,$$

where $g \in \mathcal{H}(\mathbb{D}_{r\rho})$ with $\rho > 1$. Then

$$\limsup_{n \rightarrow \infty} \sqrt[n]{E_n(\overline{B_{2,r}}, F)} \leq \frac{1}{\rho}.$$

Proof. Choose $R \in (1, \rho)$. Since g is a holomorphic function in \mathbb{D}_{rR} , we can expand g in its power series $g(z) = \sum_{k=0}^{\infty} a_k z^k$ and obtain in view of Cauchy's estimates

$$|a_k| \leq \frac{M}{(rR)^k} \quad \text{for } k \in \mathbb{N}, \quad M := \sup\{|g(z)| : |z| \leq rR\}.$$

Further, let us denote the n -th Taylor polynomial of g by $p_n(z) = \sum_{k=0}^n a_k z^k$. It follows

$$|g(z) - p_n(z)| \leq \frac{M}{R^n(R-1)} = \frac{M_1}{R^n} \quad \text{for } z \in \overline{\mathbb{D}_r}, \quad M_1 := \frac{M}{(R-1)}. \quad (2.1)$$

This estimate implies for $z \in \overline{\mathbb{D}_r}$ and n sufficiently large

$$\begin{aligned} |g(z)\overline{g(z)} - p_n(z)\overline{p_n(z)}| &\leq |g(z)\overline{g(z)} - g(z)\overline{p_n(z)}| + |g(z)\overline{p_n(z)} - p_n(z)\overline{p_n(z)}| \\ &\leq |g(z)| |\overline{g(z)} - \overline{p_n(z)}| + |\overline{p_n(z)}| |g(z) - p_n(z)| \\ &\leq \left(2|g(z)| + \frac{M_1}{R^n}\right) |g(z) - p_n(z)| \\ &\leq 3\|g\|_{\overline{\mathbb{D}_r}} \frac{M_1}{R^n}. \end{aligned}$$

Now we put similarly to [9]

$$q_0(z) := p_0(z) \quad \text{and} \quad q_k(z) := p_k(z) - p_{k-1}(z) \quad \text{for } k \in \mathbb{N},$$

and define the real-valued polynomials

$$Q_n(x, y) := \sum_{\substack{k,l=0 \\ k+l \leq n}}^n q_k(z)\overline{q_l(z)}, \quad z = x + iy, \quad n \in \mathbb{N},$$

as well as

$$P_{2n}(x, y) := \sum_{k,l=0}^n q_k(z)\overline{q_l(z)} = p_n(z)\overline{p_n(z)}, \quad z = x + iy, \quad n \in \mathbb{N}.$$

Notice,

$$P_{2n}(x, y) - Q_n(x, y) = \sum_{\substack{k,l=0 \\ k+l > n}}^n q_k(z)\overline{q_l(z)} = \sum_{k=1}^n q_k(z)(\overline{p_n(z)} - \overline{p_{n-k}(z)}).$$

Because of (2.1) we get

$$|p_l(z) - p_k(z)| \leq |g(z) - p_l(z)| + |g(z) - p_k(z)| \leq \frac{2M_1}{R^k} \quad \text{for } k < l, \quad z \in \overline{\mathbb{D}_r}.$$

Owing to the definition of q_k and the last estimate we achieve

$$|q_k(z)(\overline{p_n(z)} - p_{n-k}(z))| \leq \frac{2M_1}{R^{k-1}} \frac{2M_1}{R^{n-k}} = \frac{4M_1^2}{R^{n-1}} \quad \text{for } z \in \overline{\mathbb{D}}_r.$$

which gives

$$|P_{2n}(x, y) - Q_n(x, y)| \leq \frac{4nM_1^2}{R^{n-1}} \quad \text{for } (x, y) \in \overline{B}_{2,r}.$$

Finally we obtain

$$\begin{aligned} |F(x, y) - Q_n(x, y)| &\leq |F(x, y) - P_{2n}(x, y)| + |P_{2n}(x, y) - Q_n(x, y)| \\ &\leq 3\|g\|_{\overline{\mathbb{D}}_r} \frac{M_1}{R^n} + \frac{4nM_1^2}{R^{n-1}} \quad \text{for } (x, y) \in \overline{B}_{2,r}. \end{aligned}$$

This yields

$$\limsup_{n \rightarrow \infty} \sqrt[n]{E_n(\overline{B}_{2,r}, F)} \leq \frac{1}{\rho}$$

as $R < \rho$ was arbitrary. \square

The foundation for the upper estimate $\limsup_{n \rightarrow \infty} \sqrt[n]{E_n(\overline{B}_{2,r}, F)} \geq 1/\rho$ is laid by Lemma 1.4.

Proof of Lemma 1.4. Let $R_1 \in (1, \rho)$. Then there exists a constant $M > 0$ such that $E_n(\partial B_{2,r}, F) \leq M/R_1^n$ for all $n \in \mathbb{N}$. Since $\partial B_{2,r}$ is compact we can find a (best approximation) polynomial P_n of degree $\leq n$, $P_n : \mathbb{R}^2 \rightarrow \mathbb{R}$, to each $n \in \mathbb{N}$ satisfying

$$|F(x, y) - P_n(x, y)| \leq \frac{M}{R_1^n} \quad \text{for } (x, y) \in \partial B_{2,r}. \quad (2.2)$$

Now we define

$$p_{r,n}(z) := P_n\left(r \frac{1}{2} \left(z + \frac{1}{z}\right), r \frac{1}{2i} \left(z - \frac{1}{z}\right)\right) \quad \text{for } z \in \mathbb{C} \setminus \{0\}.$$

As

$$p_{r,n}(e^{it}) = P_n(r \cos t, r \sin t) \quad \text{for } t \in [0, 2\pi],$$

we can write (2.2) in the form

$$|F(r \cos t, r \sin t) - p_{r,n}(e^{it})| \leq \frac{M}{R_1^n} \quad \text{for } t \in [0, 2\pi].$$

Therefore we get

$$|p_{r,n+1}(z) - p_{r,n}(z)| \leq \frac{2M}{R_1^n} \quad \text{for } z \in \partial \mathbb{D},$$

which implies

$$|z^{n+1}(p_{r,n+1}(z) - p_{r,n}(z))| \leq \frac{2M}{R_1^n} \quad \text{for } z \in \partial \mathbb{D}.$$

Note, the expression on the left-hand side of the latter estimate is a complex-valued polynomial of degree $\leq 2(n+1)$. By the maximum principle we deduce

$$|z^{n+1}(p_{r,n+1}(z) - p_{r,n}(z))| \leq \frac{2M}{R_1^n} \quad \text{for } z \in \overline{\mathbb{D}}.$$

Now let R_2 be an arbitrary number of $(1, R_1)$. Then it follows

$$|p_{r,n+1}(z) - p_{r,n}(z)| \leq \frac{2M}{R_1^n} R_2^{n+1} \quad \text{for } \frac{1}{R_2} \leq |z| \leq 1.$$

Consequently, as $R_1 \in (1, \rho)$ was arbitrary, the series

$$p_{r,0} + \sum_{n=1}^{\infty} (p_{r,n} - p_{r,n-1})$$

converges locally uniformly in $\hat{A}_{1/\rho,1} := \{z \in \mathbb{C} : 1/\rho < |z| \leq 1\}$ to a function \tilde{h}_r , which is holomorphic in $A_{1/\rho,1}$ and continuous on $\hat{A}_{1/\rho,1}$.

Combining now

$$p_{r,n}(e^{it}) = P_n(r \cos t, r \sin t) \rightarrow F(r \cos t, r \sin t) \quad \text{for } n \rightarrow \infty, \quad t \in [0, 2\pi],$$

with

$$p_{r,n}(z) \rightarrow \tilde{h}_r(z) \quad \text{for } n \rightarrow \infty, \quad z \in \hat{A}_{\frac{1}{\rho},1},$$

gives

$$\tilde{h}_r(e^{it}) = F(r \cos t, r \sin t) = h_r(e^{it}) \quad \text{for } t \in [0, 2\pi].$$

Hence \tilde{h}_r is the holomorphic extension of h_r to $A_{1/\rho,1}$. Further, since the function \tilde{h}_r is continuous on $\hat{A}_{1/\rho,1}$ and real-valued on $\partial\mathbb{D}$ we can apply Schwarz's reflection principle and conclude that h_r has even a holomorphic extension to $A_{1/\rho,\rho}$. This completes the proof. \square

For technical reasons we next bring in the following notation.

Definition 2.2. If $h(z) = \sum_{k=0}^{\infty} a_k z^k$ is a holomorphic function in \mathbb{D}_r , $r > 0$, then we define $h_- \in \mathcal{H}(\mathbb{D}_r)$ by

$$h_-(z) := \sum_{k=0}^{\infty} \overline{a_k} z^k, \quad z \in \mathbb{D}_r. \quad (2.3)$$

Now we have all necessary ingredients to prove the main theorem of this paper.

Proof of Theorem 1.3. Ad (i) \Leftrightarrow (ii): Because of Lemma 2.1 we only need to verify the inequality

$$\limsup_{n \rightarrow \infty} \sqrt[n]{E_n(\overline{B}_{2,r}, F)} \geq \frac{1}{\rho} \quad (2.4)$$

if $F(x, y) = |g(x + iy)|^2$ and $g \in \mathcal{H}(\mathbb{D}_{r\rho}) \setminus \mathcal{H}(\overline{\mathbb{D}}_{r\rho})$.

Without loss of generality we may assume that $r = 1$. Otherwise consider the scaled function $\tilde{F}(x, y) = F(rx, ry)$ for $(x, y) \in B_2$.

We split the proof of estimate (2.4) into two steps. In Step 1 we handle the case that g has only zeros on $\partial\mathbb{D}$ whereas in Step 2 g oblige no “zero”-restriction.

Step 1: Our goal is to show that even the stronger estimate

$$\limsup_{n \rightarrow \infty} \sqrt[n]{E_n(\partial B_2, F)} \geq \frac{1}{\rho}$$

holds, if $F(x, y) = |g(x + iy)|^2$, $g \in \mathcal{H}(\mathbb{D}_\rho) \setminus \mathcal{H}(\overline{\mathbb{D}}_\rho)$ and g is zero-free in \mathbb{D} . In order to prove the lower bound we assume

$$\limsup_{n \rightarrow \infty} \sqrt[n]{E_n(\partial B_2, F)} \leq \frac{1}{R_1} < \frac{1}{\rho}$$

for some $R_1 > \rho$. Then by Lemma 1.4 the function

$$h_1(z) := F\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right), \quad z \in \partial\mathbb{D},$$

has a holomorphic extension \tilde{h}_1 to the annulus $A_{1/R_1, R_1}$.

Next, let a_1, \dots, a_m be the finitely many zeros of g on $\partial\mathbb{D}$. Hence we can rewrite g as

$$g(z) = \tilde{g}(z) \prod_{j=1}^m (z - a_j),$$

where $\tilde{g} \in \mathcal{H}(\mathbb{D}_\rho)$. Further, for $z \in \partial\mathbb{D}$ we get

$$\tilde{h}_1(z) = |g(z)|^2 = \tilde{g}(z) \tilde{g}_-\left(\frac{1}{z}\right) \prod_{j=1}^m (z - a_j) \left(\frac{1}{z} - \overline{a_j}\right),$$

where \tilde{g}_- is specified in Eq. (2.3). Thus

$$\tilde{h}(z) := \frac{\tilde{h}_1(z)}{\prod_{j=1}^m \left(\frac{1}{z} - \overline{a_j}\right)} = \tilde{g}(z) \tilde{g}_-\left(\frac{1}{z}\right) \prod_{j=1}^m (z - a_j)$$

is holomorphic in $A_{1/R_1, R_1}$ because $\tilde{h}_1 \in \mathcal{H}(A_{1/R_1, R_1})$.

Since \tilde{g} is zero-free in a neighborhood of $\overline{\mathbb{D}}$, we can find an $\varepsilon > 0$ with $1/R_1 < 1 - \varepsilon$ such that $\tilde{g}_-(1/z) \neq 0$ for $z \in A_{1-\varepsilon, R_1}$. Consequently,

$$\hat{h}(z) = \frac{\tilde{h}(z)}{\tilde{g}_-\left(\frac{1}{z}\right)}$$

is holomorphic in $A_{1-\varepsilon, R_1}$. The fact that \hat{h} coincides with g on $\partial\mathbb{D}$ implies that g has a holomorphic extension to \mathbb{D}_{R_1} , $R_1 > \rho$, and the contradiction is apparent.

Step 2: Here, we represent g in the form

$$g(z) = B(z) \hat{g}(z), \quad B(z) = \prod_{j=1}^m \frac{z_j - z}{1 - \overline{z_j} z}, \quad z_j \in \mathbb{D},$$

where $\hat{g} \in \mathcal{H}(\overline{\mathbb{D}})$ is zero-free in \mathbb{D} .

Now there are two possibilities for the holomorphic behavior of \hat{g} :

(a) $\hat{g} \in \mathcal{H}(\mathbb{D}_\rho) \setminus \mathcal{H}(\overline{\mathbb{D}}_\rho)$ and (b) $\hat{g} \in \mathcal{H}(\mathbb{D}_{\rho_1})$ for some $\rho_1 > \rho$.

To (a): Let $\hat{F}(x, y) := |\hat{g}(x + iy)|^2$, $(x, y) \in \overline{B}_2$. Then $F = \hat{F}$ on ∂B_2 , so

$$\limsup_{n \rightarrow \infty} \sqrt[n]{E_n(\overline{B}_2, F)} \geq \limsup_{n \rightarrow \infty} \sqrt[n]{E_n(\partial B_2, F)} = \limsup_{n \rightarrow \infty} \sqrt[n]{E_n(\partial B_2, \hat{F})} \geq \frac{1}{\rho}$$

by Step 1.

To (b): In this case, g is meromorphic in \mathbb{D}_{ρ_1} with possible poles at $1/\overline{z}_1, \dots, 1/\overline{z}_m$. As $g \in \mathcal{H}(\mathbb{D}_\rho) \setminus \mathcal{H}(\overline{\mathbb{D}}_\rho)$ we infer that $|z_l| = 1/\rho$ for at least one $l \in \{1, \dots, m\}$ such that g has a pole at $1/\overline{z}_l$. Similarly as before assume

$$\limsup_{n \rightarrow \infty} \sqrt[n]{E_n(\overline{B}_2, F)} \leq \frac{1}{R_2} < \frac{1}{\rho}$$

for some $R_2 > \rho$. Next, choose $\hat{r} \in (\rho/R_2, 1)$ such that

$$\frac{z_k}{z_l} \neq \hat{r}^2 \quad \text{for all } k \in \{1, 2, \dots, m\}.$$

Thus g_- does not vanish at $z = \hat{r}^2 z_l$. This entails that the function

$$h_{\hat{r}}(z) := g(\hat{r}z)g_- \left(\frac{\hat{r}}{z} \right)$$

has a pole at the point $z = 1/(\overline{z}_l \hat{r}) \in A_{1, R_2}$.

On the other hand, Lemma 1.4 shows that $h_{\hat{r}}$ has a holomorphic extension to $A_{1/R_2, R_2}$. This contradiction completes part (b) and therefore Step 2.

The additional statement of Theorem 1.3 is quite obvious if we regard it as the limiting case “ $\rho = \infty$ ”. \square

With the previous result we are well prepared to prove Theorem 1.5.

Proof of Theorem 1.5. Like in the previous proof we may assume that $r = 1$. Moreover, by Theorem 1.3 it suffices to check the following equivalence:

$$g \in \mathcal{H}(\mathbb{D}_\rho) \setminus \mathcal{H}(\overline{\mathbb{D}}_\rho)$$

if and only if for every $s \in (0, 1]$ the function $h_s : \partial \mathbb{D} \rightarrow \mathbb{C}$,

$$h_s(z) = F \left(s \frac{1}{2} \left(z + \frac{1}{z} \right), s \frac{1}{2i} \left(z - \frac{1}{z} \right) \right),$$

has a holomorphic extension \tilde{h}_s to $A_{s/\rho, \rho/s}$, and at least one of these extensions is not holomorphic continuable to any neighborhood of $\overline{A}_{s/\rho, \rho/s}$.

To prove the “if”-direction, let \tilde{h}_s be the holomorphic extension of h_s to $A_{\frac{s}{\rho}, \frac{\rho}{s}}$, $s \in (0, 1]$. Then, for $z \in \partial \mathbb{D}$, we have

$$|g(sz)|^2 = F(s \operatorname{Re} z, s \operatorname{Im} z) = F \left(\frac{s}{2} \left(z + \frac{1}{z} \right), \frac{s}{2i} \left(z - \frac{1}{z} \right) \right) = \tilde{h}_s(z), \quad s \in (0, 1].$$

Therefore $g(sz)$ can be represented by

$$g(sz) = \frac{\tilde{h}_s(z)}{g\left(\frac{s}{\bar{z}}\right)}$$

for all $s \in (0, 1]$ and $z \in \partial\mathbb{D}$. Since $z \mapsto \overline{g(s/\bar{z})}$ is holomorphic in $A_{s,\rho/s}$, we see that $z \mapsto g(sz)$ is for sure meromorphic in $A_{s,\rho/s}$ for each $s \in (0, 1]$. If now $z_0 \in \mathbb{D}_\rho$ is a pole of g , then $g(sz)$ has a pole at $z = z_0/s$, so $\overline{g(s/\bar{z})}$ has a zero at $z = z_0/s$ for each $s \in (0, 1]$. Thus $g(z)$ would have a zero at $z = s^2/\bar{z}_0$ for each $s \in (0, 1]$, which is clearly impossible. Consequently, $g \in \mathcal{H}(\mathbb{D}_\rho)$.

To finish the proof of the “if”-statement and to prove the “only if”-assertion let $g \in \mathcal{H}(\mathbb{D}_\rho)$. Then

$$\tilde{h}_s(z) = g(sz)g_-\left(\frac{s}{z}\right)$$

is the holomorphic extension of h_s to $A_{s/\rho,\rho/s}$ for each $s \in (0, 1]$.

A closer look at the proof of Step 2 of Theorem 1.3 shows that we find to each ε , $0 < \varepsilon < 1$, some $\hat{s} \in (1 - \varepsilon, 1]$ such that $h_{\hat{s}}$ has no holomorphic extension to any domain containing $\overline{A_{\hat{s}/\rho,\rho/\hat{s}}}$, if $g \in \mathcal{H}(\mathbb{D}_\rho) \setminus \mathcal{H}(\overline{\mathbb{D}_\rho})$. \square

3. Some applications of Theorem 1.3 and further consequences

By the maximum principle we clearly have $\|g - p\|_{\overline{\mathbb{D}_r}} = \|g - p\|_{\partial\mathbb{D}_r}$ if g and p are holomorphic functions in a neighborhood of $\overline{\mathbb{D}_r}$. Thus, for determining the complex maximal convergence number ρ for g on $\overline{\mathbb{D}_r}$ we can draw back to $\partial\mathbb{D}_r$. If now $F = |g|^2$, $g \in \mathcal{H}(\overline{\mathbb{D}_r})$, and g has no zeros in \mathbb{D}_r we get a closely tied result, see Theorem 3.1. However, in the case that F has zeros in $B_{2,r}$, Example 3.2 unveils some disparity.

Theorem 3.1. Let $F : \overline{B_{2,r}} \rightarrow \mathbb{R}$ be given by $F(x, y) = |g(x + iy)|^2$, where $g \in \mathcal{H}(\overline{\mathbb{D}_r})$.

(i) If g has either no zeros on $\overline{\mathbb{D}_r}$ or only zeros on $\partial\mathbb{D}_r$, then

$$\limsup_{n \rightarrow \infty} \sqrt[n]{E_n(\overline{B_{2,r}}, F)} = \limsup_{n \rightarrow \infty} \sqrt[n]{E_n(\partial B_{2,r}, F)} = \frac{1}{\rho}, \quad (3.1)$$

where $\rho > 1$ is the largest number such that g has a holomorphic extension to $\mathbb{D}_{r\rho}$.

(ii) If g has zeros in \mathbb{D}_r , choose the representation

$$g(z) := \hat{g}(z) \prod_{j=1}^m (z - z_j) \quad \text{for } z \in \mathbb{D}_r, \quad z_j \in \mathbb{D}_r, \quad m \in \mathbb{N},$$

where \hat{g} is a zero-free holomorphic function in \mathbb{D}_r . Further, define $\hat{F} : \overline{B_{2,r}} \rightarrow \mathbb{R}$ by

$$\hat{F}(x, y) = |\hat{g}(x + iy)|^2.$$

Then

$$\limsup_{n \rightarrow \infty} \sqrt[n]{E_n(\overline{B_{2,r}}, F)} = \limsup_{n \rightarrow \infty} \sqrt[n]{E_n(\partial B_{2,r}, \hat{F})} = \frac{1}{\rho},$$

where $\rho > 1$ is the largest number such that g has a holomorphic extension to $\mathbb{D}_{r\rho}$.

Proof.

To (i): Since g is holomorphic in $\mathcal{H}(\mathbb{D}_{r\rho}) \setminus \mathcal{H}(\overline{\mathbb{D}}_{r\rho})$ we derive from Theorem 1.3

$$\limsup_{n \rightarrow \infty} \sqrt[n]{E_n(\overline{B}_{2,r}, F)} = \frac{1}{\rho}.$$

By Step 1 of the proof of Theorem 1.3 we have the relation

$$\frac{1}{\rho} = \limsup_{n \rightarrow \infty} \sqrt[n]{E_n(\overline{B}_{2,r}, F)} = \limsup_{n \rightarrow \infty} \sqrt[n]{E_n(\partial B_{2,r}, F)}.$$

To (ii): Again from Theorem 1.3 we conclude

$$\limsup_{n \rightarrow \infty} \sqrt[n]{E_n(\overline{B}_{2,r}, F)} = \frac{1}{\rho}.$$

Since $\hat{F} = |\hat{g}|^2$, \hat{g} is zero-free in \mathbb{D}_r and $\hat{g} \in \mathcal{H}(\mathbb{D}_{r\rho}) \setminus \mathcal{H}(\overline{\mathbb{D}}_{r\rho})$, we obtain

$$\frac{1}{\rho} = \limsup_{n \rightarrow \infty} \sqrt[n]{E_n(\overline{B}_{2,r}, F)} = \limsup_{n \rightarrow \infty} \sqrt[n]{E_n(\overline{B}_{2,r}, \hat{F})} = \limsup_{n \rightarrow \infty} \sqrt[n]{E_n(\partial B_{2,r}, \hat{F})}. \quad \square$$

Our next example illustrates that Eq. (3.1) fails if g has zeros in \mathbb{D}_r .

Example 3.2. Let F be the squared modulus of a Blaschke product, i.e.

$$F(x, y) = \left| \prod_{j=1}^m \frac{z_j - z}{1 - \overline{z}_j z} \right|^2 \quad \text{for } (x, y) \in \overline{B}_2, \quad z = x + iy, \quad z_j \in \mathbb{D}, \quad m \in \mathbb{N}.$$

Then

$$\limsup_{n \rightarrow \infty} \sqrt[n]{E_n(\partial B_2, F)} = 0$$

but

$$\limsup_{n \rightarrow \infty} \sqrt[n]{E_n(\overline{B}_2, F)} = \max_{1 \leq j \leq m} |z_j|.$$

A natural question which may arise is, whether a similar maximal convergence result like Theorem 1.3 also holds for functions of the form

$$F(x, y) = |g(x + iy)|, \quad g \in \mathcal{H}(\overline{\mathbb{D}}_r).$$

Obviously, if F is a zero-free function defined on $\overline{B}_{2,r}$ we receive for the maximal convergence number $\rho^{1/2}$ in the case that $g \in \mathcal{H}(\mathbb{D}_{r\rho}) \setminus \mathcal{H}(\overline{\mathbb{D}}_{r\rho})$. However, the situation is different if F has zeros in $\overline{B}_{2,r}$. Theorem 3.3 reveals that then there exists no sequence of polynomials which converges maximally to F .

Theorem 3.3. Let $g \in \mathcal{H}(\overline{\mathbb{D}}_r)$ and $F : \overline{B}_{2,r} \rightarrow \mathbb{R}$ be defined by

$$F(x, y) = |g(x + iy)|^2 \prod_{j=1}^m |x + iy - a_j|, \quad a_j \in \overline{\mathbb{D}}_r, \quad a_l \neq a_k \text{ for } l \neq k.$$

Then

$$\limsup_{n \rightarrow \infty} \sqrt[n]{E_n(\overline{B}_{2,r}, F)} > \frac{1}{\rho} \quad \text{for any } \rho > 1. \quad (3.2)$$

Proof. The statement is proved by contradiction. Without loss of generality we may assume $r = 1$. Next we distinguish the cases (a) $a_k \neq 0$ for some $k \in \{1, \dots, m\}$ and (b) $m = 1$ and $a_m = 0$.

To (a): We assume there exists a function of the form

$$F(x, y) = |g(x + iy)|^2 \prod_{j=1}^m |x + iy - a_j|, \quad g \in \mathcal{H}(\overline{\mathbb{D}}), \quad a_j \in \overline{\mathbb{D}}, \quad a_l \neq a_k \text{ for } l \neq k,$$

which can be approximated maximally by polynomials. Then

$$\limsup_{n \rightarrow \infty} \sqrt[n]{E_n(\overline{B}_2, F)} \leq \frac{1}{\tilde{\rho}}$$

for some $\tilde{\rho} \in (1, \hat{\rho})$, where $\hat{\rho} > 1$ is chosen so small that g is also holomorphic in $\mathbb{D}_{\hat{\rho}}$.

Due to Lemma 1.4 we know that each function $h_s : \partial\mathbb{D} \rightarrow \mathbb{C}$, $h_s(z) = |g(sz)|^2 \prod_{j=1}^m |sz - a_j|$, $s \in (0, 1]$, has a holomorphic extension \tilde{h}_s to $A_{1/\rho, \rho}$.

For $z \in \partial\mathbb{D}$ we have

$$\begin{aligned} \tilde{h}_s(z) &= F\left(s\frac{1}{2}\left(z + \frac{1}{z}\right), s\frac{1}{2i}\left(z - \frac{1}{z}\right)\right) \\ &= g(sz)g_{-}\left(s\frac{1}{z}\right) \prod_{j=1}^m \sqrt{\left(s\frac{1}{2}\left(z + \frac{1}{z}\right) - \operatorname{Re} a_j\right)^2 + \left(s\frac{1}{2i}\left(z - \frac{1}{z}\right) - \operatorname{Im} a_j\right)^2} \\ &= \tilde{g}(sz)\tilde{g}_{-}\left(s\frac{1}{z}\right) \prod_{k=1}^l \left((sz - b_k)\left(s\frac{1}{z} - \bar{b}_k\right)\right) \\ &\quad \times \prod_{j=1}^m \sqrt{\left(s\frac{1}{2}\left(z + \frac{1}{z}\right) - \operatorname{Re} a_j\right)^2 + \left(s\frac{1}{2i}\left(z - \frac{1}{z}\right) - \operatorname{Im} a_j\right)^2}, \end{aligned}$$

where $g(z) = \tilde{g}(z) \prod_{k=1}^l (z - b_k)$, $\tilde{g}(z) \neq 0$ for $z \in \overline{\mathbb{D}}$, $b_k \in \overline{\mathbb{D}}$ and \tilde{g}_{-} as well as g_{-} are defined as in Eq. (2.3). Now we choose $\varepsilon > 0$ so small that $\tilde{g} \in \mathcal{H}(\mathbb{D}_{1+\varepsilon})$ and $A_{1-\varepsilon, 1+\varepsilon} \subset A_{1/\tilde{\rho}, \tilde{\rho}}$. In addition, we also may assume that $\tilde{g}(z) \neq 0$ and $\tilde{g}_{-}(1/z) \neq 0$ for $z \in A_{1-\varepsilon, 1+\varepsilon}$.

Thus the function $l_s : \partial\mathbb{D} \rightarrow \mathbb{C}$, $s \in (0, 1]$, defined by

$$\begin{aligned} l_s(z) &= \prod_{k=1}^l \left((sz - b_k)\left(s\frac{1}{z} - \bar{b}_k\right)\right) \\ &\quad \times \prod_{j=1}^m \sqrt{\left(s\frac{1}{2}\left(z + \frac{1}{z}\right) - \operatorname{Re} a_j\right)^2 + \left(s\frac{1}{2i}\left(z - \frac{1}{z}\right) - \operatorname{Im} a_j\right)^2} \end{aligned}$$

has a holomorphic extension to $A_{1-\varepsilon, 1+\varepsilon}$, because the functions

$$\tilde{l}_s(z) := \frac{\tilde{h}_s(z)}{\tilde{g}(sz)\tilde{g}_-\left(s\frac{1}{z}\right)}, \quad s \in (0, 1],$$

are holomorphic in $A_{1-\varepsilon, 1+\varepsilon}$ and $\tilde{l}_s \equiv l_s$ on $\partial\mathbb{D}$.

However, if $s_k = |a_k|$ and $|a_k| > 0$, $k \in \{1, \dots, m\}$, then the function l_{s_k} has a branch point on $\partial\mathbb{D}$. Hence it can not be holomorphic in $A_{1-\varepsilon, 1+\varepsilon}$ and the result follows.

To (b): In this case F has the representation

$$F(x, y) = |g(z)|^2|z|, \quad g \in \mathcal{H}(\overline{\mathbb{D}}), \quad z = x + iy.$$

Therefore let us consider a restriction of F . We define $\tilde{F} : \overline{B}_{2, 1-a} \rightarrow \mathbb{R}$, $0 < a < 1/2$, by

$$\tilde{F}(x, y) = |g(z-a)|^2|z-a|.$$

For \tilde{F} we can apply similar arguments as in (a) if we replace \overline{B}_2 by $\overline{B}_{2, 1-a}$ and $r \in (0, 1]$ by $r \in (0, 1-a)$. We obtain finally that

$$l_a(z) = |a| \sqrt{\left(\frac{1}{2} \left(z + \frac{1}{z}\right) - 1\right)^2 + \left(\frac{1}{2i} \left(z - \frac{1}{z}\right)\right)^2}, \quad z \in \mathbb{D},$$

has a holomorphic extension to $A_{1-\varepsilon, 1+\varepsilon}$, which is absurd.

As \tilde{F} is a restriction of F to a subset of \overline{B}_2 the proof is finished. \square

Lemma 1.4 is a powerful tool to determine upper bounds for E_n even for non-squared modulus holomorphic functions. An application of it is shown in the next corollary.

Corollary 3.4. *Let*

$$F(x, y) = \frac{1}{a - xy} \quad \text{for } (x, y) \in \overline{B}_2, \quad a \in \mathbb{R} \setminus [-1, 1],$$

then

$$\limsup_{n \rightarrow \infty} \sqrt[n]{E_n(\overline{B}_2, F)} = \frac{1}{\rho},$$

where $\rho > 1$ is uniquely determined by

$$2|a| = \frac{1}{2} \left(\rho^2 + \frac{1}{\rho^2} \right).$$

Proof.

“ \geq ”: We plug $x = \frac{1}{2}(z + \frac{1}{z})$ and $y = \frac{1}{2i}(z - \frac{1}{z})$ in F and define the function

$$\tilde{h}_1(z) := \frac{1}{a - \frac{1}{4i} \left(z^2 - \frac{1}{z^2} \right)},$$

which is holomorphic in \mathbb{C} except at the points z_j , $j \in \{1, 2, 3, 4\}$, where

$$a = \frac{1}{4i} \left(z_j^2 - \frac{1}{z_j^2} \right).$$

Now let us set $\rho := \min\{|z_j| : |z_j| > 1, j \in \{1, 2, 3, 4\}\}$. Then Lemma 1.4 implies

$$\limsup_{n \rightarrow \infty} \sqrt[n]{E_n(\overline{B}_2, F)} \geq \frac{1}{\rho}.$$

“ \leq ”: Let us consider the function $G : [-1, 1] \rightarrow \mathbb{R}$ defined by

$$G(u) = \frac{2}{2a - u}.$$

From Theorem 1.1 we know that there exists a sequence of polynomials P_n of degree $\leq n$ satisfying

$$|G(u) - P_n(u)| \leq \frac{M}{R^n} \quad \text{for all } n \in \mathbb{N}, \quad u \in [-1, 1],$$

where $M > 0$ is some constant independent of n , R is any number of the interval $(1, \rho_1)$ and $\rho_1 > 1$ is uniquely determined by

$$2|a| = \frac{1}{2} \left(\rho_1 + \frac{1}{\rho_1} \right).$$

Notice, if $(x, y) \in \overline{B}_2$ we have $2xy \in [-1, 1]$ and therefore

$$F(x, y) = \frac{1}{a - xy} = \frac{2}{2a - 2xy} = \frac{2}{2a - u} = G(u) \quad \text{for } u = 2xy, \quad (x, y) \in \overline{B}_2.$$

Hence we get

$$|F(x, y) - \tilde{P}_{2n}(x, y)| \leq \frac{M}{R^n} \quad \text{for } (x, y) \in \overline{B}_2,$$

where $\tilde{P}_{2n}(x, y) := P_n(2xy)$. As \tilde{P}_{2n} is a polynomial of degree $\leq 2n$ we achieve

$$\limsup_{n \rightarrow \infty} \sqrt[n]{E_n(\overline{B}_2, F)} \leq \frac{1}{\sqrt{\rho_1}}.$$

Because of $\left(\rho_1 + \frac{1}{\rho_1}\right) = \frac{1}{i} \left(i\rho_1 - \frac{1}{i\rho_1}\right)$ we obtain $\sqrt{\rho_1} = \rho$, which completes the proof. \square

4. Maximal convergence results for squared holomorphic functions on arbitrary compact sets

In this section we investigate the connection between the real maximal convergence number and the corresponding complex maximal convergence number for squared holomorphic functions on arbitrary sufficiently nice compact sets. From the proof of Lemma 2.1 we can easily extract:

Corollary 4.1. Let $K \subset \mathbb{C}$ be a compact set and g be a holomorphic function in some open connected neighborhood of K . Further, let the function F be given by

$$F(x, y) = |g(x + iy)|^2 \quad \text{for } (x, y) \in L := \{(\operatorname{Re} z, \operatorname{Im} z) : z \in K\}.$$

If there exists a sequence $\{p_n\}_{n \in \mathbb{N}}$ of complex-valued polynomials p_n of degree $\leq n$ such that

$$|g(z) - p_n(z)| \leq \frac{M}{R^n}, \quad z \in K, \quad n \in \mathbb{N},$$

for some $R \in (1, \infty)$ and some constant $M > 0$ independent of n , then

$$\limsup_{n \rightarrow \infty} \sqrt[n]{E_n(L, F)} \leq \frac{1}{R}.$$

If we now combine the latter result with Theorem 1.2 we get the following statement.

Corollary 4.2. Let K be a compact subset of \mathbb{C} , such that $\hat{\mathbb{C}} \setminus K$ is connected and regular. Further, let F be given by

$$F(x, y) = |g(x + iy)|^2 \quad \text{for } (x, y) \in L := \{(\operatorname{Re} z, \operatorname{Im} z) : z \in K\},$$

where $g \in \mathcal{H}(L_\rho)$, $L_\rho := \{z \in \mathbb{C} : e^{g_K(z)} < \rho\}$, and g_K is Green's function for $\hat{\mathbb{C}} \setminus K$ with pole at infinity. Then

$$\limsup_{n \rightarrow \infty} \sqrt[n]{E_n(L, F)} \leq \frac{1}{\rho}. \quad (4.1)$$

However we do not get the opposite direction of Corollary 4.2 in general. This fact is illustrated in the next theorem for a closed square.

Theorem 4.3. Consider the function

$$F(x, y) = \frac{1}{((x - \rho_1)^2 + y^2)^s} \quad \text{for } (x, y) \in [-1, 1] \times [-1, 1],$$

where $s \in (0, \infty)$ and $\rho_1 \in (1, \infty)$. Further, define the function

$$g(z) = \frac{1}{(z - \rho_1)^s} \quad \text{for } z \in K := \{z \in \mathbb{C} : z = x + iy, \ x, y \in [-1, 1]\}.$$

Then

$$\limsup_{n \rightarrow \infty} \sqrt[n]{E_n([-1, 1] \times [-1, 1], F)} = \frac{1}{\rho_1} \quad (4.2)$$

but

$$\limsup_{n \rightarrow \infty} \sqrt[n]{e_n(K, g)} = \frac{1}{|\psi(\rho_1)|} > \frac{1}{\rho_1}, \quad (4.3)$$

where ψ maps $\hat{\mathbb{C}} \setminus K$ univalently onto $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ such that $\psi(\infty) = \infty$.³

³ The conformal mapping ψ is up to a rotation uniquely determined.

However, before we prove Theorem 4.3 we have to verify two auxiliary results about conformal mappings.

Lemma 4.4. *Let $K \subset \mathbb{C}$, $K \neq \emptyset$, be compact such that $\hat{\mathbb{C}} \setminus K$ is simply connected and let $g : K \rightarrow \mathbb{C}$ be a continuous function. Furthermore, let $\rho > 1$. Then*

$$\limsup_{n \rightarrow \infty} \sqrt[n]{e_n(K, g)} \leq \frac{1}{\rho}$$

if and only if g has a holomorphic extension to the set $K \cup \{z \in \mathbb{C} : 1 < |\psi(z)| < \rho\}$, where ψ is a function which maps $\hat{\mathbb{C}} \setminus K$ univalently onto $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ such that $\psi(\infty) = \infty$.

Proof. This result is an immediate consequence of Theorem 1.2. We only have to take into account that for simply connected proper subsets of $\hat{\mathbb{C}}$ Green's function for $\hat{\mathbb{C}} \setminus K$ with pole at infinity coincides with $\log |\psi|$ on $\hat{\mathbb{C}} \setminus K$. \square

Lemma 4.5. *Let $K \subset \mathbb{C}$ be a compact set with $\overline{\mathbb{D}} \not\subseteq K$. If there exists a function ψ which maps $\hat{\mathbb{C}} \setminus K$ univalently onto $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ such that $\psi(\infty) = \infty$, then*

$$|\psi(z)| < |z| \quad \text{for } z \in \mathbb{C} \setminus K.$$

Proof. Consider the function $h : \mathbb{D} \rightarrow \mathbb{C}$ defined by

$$h(z) = \frac{1}{\psi^{-1}\left(\frac{1}{z}\right)},$$

where ψ^{-1} is the inverse function of ψ . Then h is holomorphic in \mathbb{D} . Moreover, we have $h(\mathbb{D}) \subset \mathbb{D}$ and $h(0) = 0$. Hence by Schwarz's Lemma we obtain that

$$\left| \psi^{-1}\left(\frac{1}{z}\right) \right| > \left| \frac{1}{z} \right| \quad \text{for } z \in \mathbb{D} \setminus \{0\},$$

and therefore

$$|\psi(z)| < |z| \quad \text{for } z \in \mathbb{C} \setminus K. \quad \square$$

To prove Theorem 4.3 we also make use of a theorem due to Sapogov [16] which is the analogue of Bernstein's theorem in higher dimensions. For our considerations it suffices to formulate this theorem for the two-dimensional case.

Theorem 4.6 (cf. Sapogov [16]). *Let $F : K \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ be a continuous function, where $K := [-1, 1] \times [-1, 1]$ and $\rho > 1$.*

Then

$$\limsup_{n \rightarrow \infty} \sqrt[n]{E_n(K, F)} \leq \frac{1}{\rho}$$

if and only if F has a holomorphic extension to

$$L_\rho \times L_\rho,$$

where $L_\rho = \{z \in \mathbb{C} : |h(z)| < \rho\}$ and h is defined as in Theorem 1.1.

Proof of Theorem 4.3. Eq. (4.3) is an immediate consequence of Lemmas 4.4 and 4.5. Thus it remains to prove Eq. (4.2).

“ \leq ”: Theorem 4.6 yields

$$\limsup_{n \rightarrow \infty} \sqrt[n]{E_n([-1, 1] \times [-1, 1], F)} \leq \frac{1}{\rho}$$

if and only if F has an analytic continuation to

$$\left\{ (z_1, z_2) \in \mathbb{C}^2 : \max \left\{ \left| z_1 + \sqrt{z_1^2 - 1} \right|, \left| z_2 + \sqrt{z_2^2 - 1} \right| \right\} < \rho \right\},$$

where the branch of the square root is chosen such that $\sqrt{z} > 0$ for $z > 0$.

Now, F has a unique holomorphic extension to $\mathbb{C}^2 \setminus \{(z_1, z_2) \in \mathbb{C}^2 : z_2 = \pm i(z_1 - \rho_1)\}$ with non-removable singularities at $z_2 = \pm i(z_1 - \rho_1)$, where $z_1 \in \mathbb{C}$ is arbitrary.

Therefore we have to show that these singularities fulfill the condition

$$\max \left\{ \left| z_1 + \sqrt{z_1^2 - 1} \right|, \left| z_2 + \sqrt{z_2^2 - 1} \right| \right\} \geq \rho_1. \quad (4.4)$$

For that reason we write z_1 in the form

$$z_1 = \frac{1}{2} \left(R + \frac{1}{R} \right) \cos t + i \frac{1}{2} \left(R - \frac{1}{R} \right) \sin t, \quad R \in [1, \infty), \quad t \in [0, 2\pi]. \quad (4.5)$$

If $R \geq \rho_1$ then inequality (4.4) is obviously satisfied, as $|z_1 + \sqrt{z_1^2 - 1}| = R$. Hence we only need to verify that

$$\left| z_2 + \sqrt{z_2^2 - 1} \right| \geq \rho_1$$

if $R \in [1, \rho_1)$. Because of Eq. (4.5) we have for z_2 the representation

$$z_2 = \pm \left(-\frac{1}{2} \left(R - \frac{1}{R} \right) \sin t + i \left(\frac{1}{2} \left(R + \frac{1}{R} \right) \cos t - \rho_1 \right) \right).$$

The modulus of the imaginary part of z_2 can be estimated by

$$\left| \frac{1}{2} \left(R + \frac{1}{R} \right) \cos t - \rho_1 \right| \geq \rho_1 - \frac{1}{2} \left(R + \frac{1}{R} \right) > \frac{1}{2} \left(\rho_1 - \frac{1}{\rho_1} \right).$$

Now, bearing the mapping properties of the inverse Joukowski function in mind, we obtain

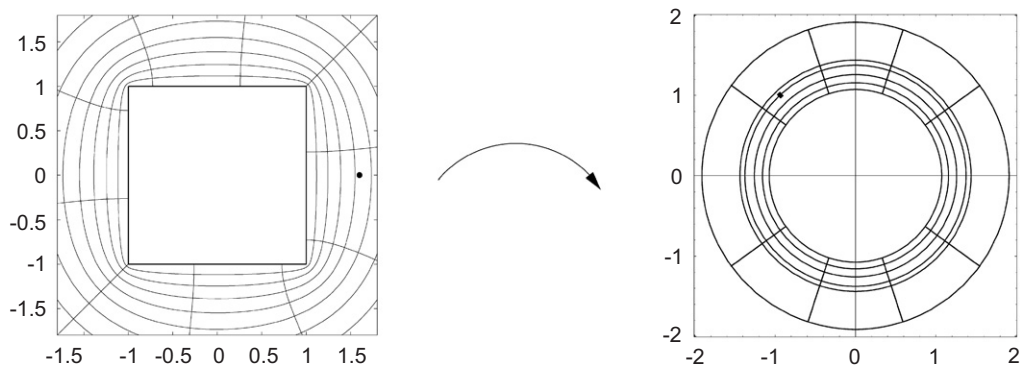
$$\left| z_2 + \sqrt{z_2^2 - 1} \right| > \rho_1.$$

Consequently,

$$\limsup_{n \rightarrow \infty} \sqrt[n]{E_n([-1, 1] \times [-1, 1], F)} \leq \frac{1}{\rho_1}.$$

“ \geq ”: This direction follows from Theorem 1.3. Since $[-1, 1] \times [-1, 1] \supset B_2$ we have

$$\limsup_{n \rightarrow \infty} \sqrt[n]{E_n([-1, 1] \times [-1, 1], F)} \geq \limsup_{n \rightarrow \infty} \sqrt[n]{E_n(B_2, F)} \geq \frac{1}{\rho_1}. \quad \square$$

Fig. 1. $\psi(z)$.

Remark 4.7. The conformal mapping ψ of Theorem 4.3 takes the form

$$\psi(z) = e^{i\varphi} \left(\sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \frac{1}{4n-1} \right)^{-1} \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} \frac{1}{4n-1} z^{-4n+1}, \quad \varphi \in [0, 2\pi],$$

see for example [12] for the construction of ψ .

If $\rho_1 = 1.4$ then we compute $\rho = |\psi(\rho_1)| = 1.26540$ up to five digits accuracy,⁴ see Fig. 1. Theorems 4.2 and 4.3 lead to the open question.

Problem 4.8. For which compact sets $K \subset \mathbb{C}$ can the following statement be confirmed.

Let K be a compact subset of \mathbb{C} , such that $\hat{\mathbb{C}} \setminus K$ is connected and regular. Furthermore, let $L := \{(\operatorname{Re} z, \operatorname{Im} z) : z \in K\}$ and $F : L \rightarrow \mathbb{R}$ be defined by

$$F(x, y) = |g(x + iy)|^2,$$

where g is holomorphic in a neighborhood of K . Then

$$\limsup_{n \rightarrow \infty} \sqrt[n]{E_n(L, F)} = \frac{1}{\rho}$$

if and only if $g \in \mathcal{H}(L_\rho) \setminus \mathcal{H}(\overline{L}_\rho)$, where $L_\rho := \{z \in \mathbb{C} : e^{g_K(z)} < \rho\}$ and g_K is Green's function for $\hat{\mathbb{C}} \setminus K$ with pole at infinity.

⁴ The plot of the mapping was produced by using the Schwarz–Christoffel Toolbox for Matlab. This toolbox is especially then helpful if one is interested in the explicit maximal convergence number ρ for a holomorphic function defined on a polygon in \mathbb{C} as it can compute Schwarz–Christoffel mappings to eight digits accuracy if crowding does not become severe [11].

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